Tutorial 3: Your Very Own EigenSolver

The purpose of this tutorial is to build a polynomial systems solver which is based on the eigenvector method. We continue to work over $K = \mathbb{Q}$ and let $P = K[x_1, \dots, x_n]$. Given polynomials $f_1, \dots, f_s \in P \setminus \{0\}$ which generate a zero-dimensional ideal $I = \langle f_1, \dots, f_s \rangle$, we let $A = P/I$.

a) Write a CoCoA function `MultMat(...)` which takes I and $i \in \{1, \dots, n\}$ and computes the matrix of the multiplication map $\mu_{x_i} : A \longrightarrow A$ with respect to the basis $\mathcal{O}_\sigma(I) = \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$. (Here σ denotes the current term ordering.)

   Hint: The CoCoA command `QuotientBasis(...)` may come in handy.

b) Compute all multiplication matrices for the following zero-dimensional ideals. Use $\sigma =$ `DegRevLex`.

   1. $I_1 = \langle 2x^2 + 3xy + y^2 - 3x - 3y,\ xy^2 - x,\ y^3 - y \rangle \subseteq \mathbb{Q}[x, y]$
   2. $I_2 = \langle 3x^2 + 4xy + y^2 - 7x - 5y + 4,\ 3y^3 + 10xy + 7y^2 - 4x - 20y + 4,\ 3xy^2 - 7xy - 7y^2 - 2x + 11y + 2 \rangle \subseteq \mathbb{Q}[x, y]$
   3. $I_3 = \langle x^2 - 2xz + 5,\ xy^2 + yz + 1,\ 3y^2 - 8xz \rangle \subseteq \mathbb{Q}[x, y, z]$
   4. $I_4 = \langle x^2 + 2y^2 - y - 2z,\ x^2 - 8y^2 + 10z - 1,\ x^2 - 7yz \rangle \subseteq \mathbb{Q}[x, y, z]$

c) Prove that a matrix $M \in \mathrm{Mat}_d(K)$ is non-derogatory if and only if the matrices $I_d, M, M^2, \dots, M^{d-1}$ are K-linearly independent. Here I_d denotes the identity matrix of size $d \times d$.

d) Implement a CoCoA function `IsNonDerogatory(...)` which takes a matrix $M \in \mathrm{Mat}_d(K)$, checks whether M is non-derogatory and returns the corresponding Boolean value.

   Hint: You may want to use `Flatten(...)` and `Syz(...)`.

e) Use your function `IsNonDerogatory(...)` to check which of the multiplication matrices in (b) are non-derogatory.

f) Implement a CoCoA function `EigenSolver(...)` which takes a zero-dimensional ideal $I = \langle f_1, \dots, f_s \rangle \subseteq P$ and performs the following operations.

   1. Replace I by its radical $\sqrt{I}$ (see Tutorial 1).
   2. Compute the multiplication matrices $M_1, \dots, M_n$ of $A = P/I$ w.r.t. the K-basis $\mathcal{O}_\sigma(I)$. Using `Sorted(...)`, order this basis increasingly w.r.t. σ.
   3. Check whether one of the multiplication matrices is non-derogatory. If this is not the case, abort with an error message. Otherwise, let M_i be non-derogatory.
   4. Compute the eigenspaces of $(M_i)^{\mathrm{tr}}$. Let $v_1, \dots, v_d$ be vectors which span the eigenspaces and have first coordinate 1. Here $d = \dim_K(A)$.

Hint: If it works, the CoCoA function `Eigenvectors(...)` can be used here. You can also program a function `CharPoly(...)`, use `Factor(...)`, and find the eigenspaces of the K-rational eigenvalues using `LinKer(...)`.

5. Read the coordinates of the solution points off the vectors $v_1, \dots, v_d$.

g) Apply your `EigenSolver(...)` to the ideals $I_1, \dots, I_4$. How many rational solutions can you find?

h) $(*)$ If you are adventurous, you may extend your `Eigensolver(...)` by treating the non-rational eigenvalues. For the real ones you can approximate them with `RealRoots(...)` and compute the eigenspaces of the approximations with the numerical functions of ApCoCoA. In the general case, you can mimic the computation over the number field $\mathbb{Q}[t]/\langle f(t)\rangle$ where $f(t)$ is the minimal polynomial of the eigenvalue.