Tutorial 6: An Approximate Tutorial

This last tutorial is meant to give you some approximate insight into the ABM-Algorithm. Let us begin by recalling this algorithm.

Let $P = \mathbb{R}[x_1, \dots, x_n]$, let $\mathbb{X} = \{p_1, \dots, p_s\} \subseteq [-1,1]^n$, let σ be a degree compatible term ordering, and let $\varepsilon > \varepsilon' > 0$.

1. Let $G = \emptyset$, $\mathcal{O} = \{1\}$, $\mathcal{M} = (1, \dots, 1)$, and $d = 0$.
2. Increase d by one. Let $L = [t_1, \dots, t_\ell]$ be $\mathbb{T}^n_d \setminus \langle \mathrm{LT}_\sigma(G) \rangle$, ordered decreasingly w.r.t. σ. If $L = \emptyset$, return $(G, \mathcal{O})$ and stop.
3. Append $\mathrm{eval}(t_1), \dots, \mathrm{eval}(t_\ell)$ as new first rows to $\mathcal{M}$ and get a matrix $\mathcal{A}$. Using the SVD of $\mathcal{A}^{\mathrm{tr}}$, compute a matrix $\mathcal{B}$ whose rows are a basis of $\mathrm{apker}(\mathcal{A}^{\mathrm{tr}}, \varepsilon)$.
4. Reduce $\mathcal{B}$ to row echelon form. Normalize each row after every reduction step. If at some point a column contains no pivot element of absolute value $> \varepsilon'$ in the untreated rows, replace the corresponding elements by zero. The result is a matrix $\mathcal{C} = (c_{ij})$.
5. For the columns j of $\mathcal{C}$ containing a pivot element c_{ij}, append the polynomial corresponding to row i to G.
6. For the columns j of $\mathcal{C}$ containing no pivot element, append t_j to $\mathcal{O}$, append the row $\mathrm{eval}(t_j)$ as a new first row to $\mathcal{M}$, and continue with (2).

This is an algorithm which computes a pair $(G, \mathcal{O})$. The list G is a unitary minimal σ-Gröbner basis of the ideal $I = \langle G \rangle \subset P$. The ideal I is zero-dimensional and a δ-approximate vanishing ideal of $\mathbb{X}$. The list $\mathcal{O}$ contains an order ideal of monomials whose residue classes form an $\mathbb{R}$-vector space basis of P/I.

In the following we will use ApCoCoA to work through one example for this algorithm.

a) Let $\mathbb{X}_1 = \{(0.01, 0.01),\ (0.49, 0),\ (0.51, 0),\ (0, 0.99)\}$, use the threshold numbers $\varepsilon = 0.1$ and $\varepsilon' = 10^{-6}$, and let $\sigma =$ `DegRevLex`.

   Show that in degree $d = 1$ we have the singular values $s_1 = 2.13$, $s_2 = 0.91$ and $s_3 = 0.35$. Hence no singular value trunction is necessary. Furthermore, show that $\mathcal{B} = \mathcal{C} = (0, 0, 0)$.

b) Show that in degree $d = 2$ we have the singular values $s_1 = 2.22$, $s_2 = 1.21$, $s_3 = 0.40$, and $s_4 = 0.006$. Perform the singular value truncation and compute the matrix $\widetilde{\mathcal{A}}$. Compare $\widetilde{\mathcal{A}}$ to the original matrix $\mathcal{A}$.

   Hint: The CoCoA command `FloatStr(...)` is useful here.

c) Prove that the space $\mathrm{apker}(\mathcal{A}^{\mathrm{tr}}, \varepsilon)$ is generated by the rows of

$$\mathcal{B} = \begin{pmatrix} 0.65 & -0.66 & 0.08 & -0.33 & -0.08 & 0.004 \\ 0.07 & -0.10 & -0.70 & -0.02 & 0.70 & -0.007 \\ 0.60 & 0.74 & -0.02 & -0.30 & 0.02 & 0.003 \end{pmatrix}$$

d) Write a CoCoA function `Normalize(...)` which takes a row vector $v = (v_1, \dots, v_n)$ and computes an approximation of its Euclidean norm $\|v\|$ in the following way. First multiply the numerator and the denominator of $c = v_1^2 + \cdots + v_n^2$ by a sufficiently large integer (e.g. by 10^{100}). Then take `Isqrt(...)` of both integers and form the quotient of the results.

e) Using your function `Normalize(...)`, perform a **normalized** Gaußian reduction on $\mathcal{B}$. Show that the result is the matrix

$$\mathcal{C} = \begin{pmatrix} 0.65 & -0.66 & 0.08 & -0.33 & -0.08 & 0.004 \\ 0 & -0.027 & -0.707 & 0.014 & 0.707 & -0.007 \\ 0 & 0 & -0.707 & 0.014 & 0.707 & -0.007 \end{pmatrix}$$

f) Conclude that a δ-approximate vanishing ideal of $\mathbb{X}_1$ is given by $I = \langle g_1, g_2, g_3 \rangle$, where $g_1 = 0.65x^2 - 0.66xy + 0.08y^2 - 0.33x - 0.08y + 0.004$, $g_2 = -0.027xy - 0.707y^2 + 0.014x + 0.707y - 0.007$, and $g_3 = -0.707y^2 + 0.014x + 0.707y - 0.007$. (Can you find an approximation for δ?) Interpret the result.

g) Using ApCoCoA interactively in a similar way, apply the ABM-Algorithm to the following cases and interpret the results. Use $\varepsilon = 0.1$ and $\varepsilon' = 10^{-6}$ again, and let $\sigma =$ `DegRevLex`.

1. $\mathbb{X}_2 = \{(0.302, 0.399),\ (0.001, 0.103), (-0.405, 0.297),\ (0.296, -0.398),\ (-0.002, -0.096)\}$
2. $\mathbb{X}_3 = \{(0.25, 0.102), (0.3, 0.101),\ (-0.095, 0.102),\ (-0.101, 0.098),\ (0.76, 0.097),\ (0.81, 0.095)\}$

h) Now implement the ABM-Algorithm in a CoCoA function `ABM(...)`. The function should take $\mathbb{X}$, ε and ε' and return a Gröbner basis of a δ-approximate vanishing ideal I of $\mathbb{X}$ with respect to the current term ordering, an order ideal $\mathcal{O}$ which is an $\mathbb{R}$-basis for P/I, and an approximation for δ.

Hint: You may want to write an auxiliary function `NormalizedGauss(...)` first. The CoCoA data type of a record can be used to collect information of very different types (such as I, $\mathcal{O}$, and δ) in one object.

i) Apply your function `ABM(...)` to the sets $\mathbb{X}_1, \mathbb{X}_2$, and $\mathbb{X}_3$ above and compare the results to your interactive computation.

... and this is approximately

The End